SU(2) gauge fields and quantum entanglement

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Based mainly on:
J. M. and T. Trześniewski,
Gauge fields and quantum entanglement,
arXiv:1911.10208

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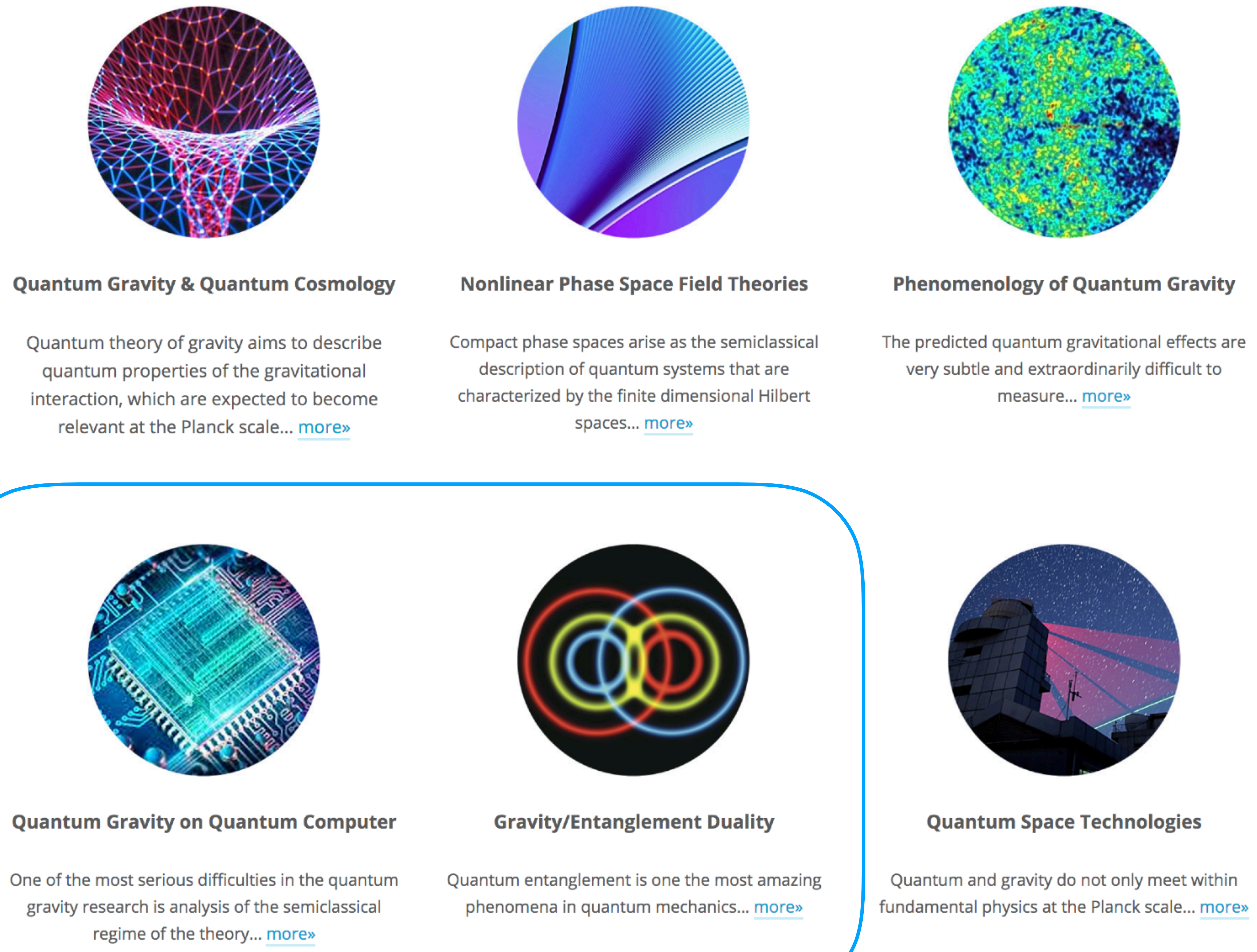


Quantum Gravity & Quantum Cosmology

Quantum theory of gravity aims to describe quantum properties of the gravitational interaction, which are expected to become relevant at the Planck scale... more»

Nonlinear Phase Space Field Theories

Compact phase spaces arise as the semiclassical description of quantum systems that are characterized by the finite dimensional Hilbert spaces... more»

Phenomenology of Quantum Gravity

The predicted quantum gravitational effects are very subtle and extraordinarily difficult to measure... more»

Quantum Gravity on Quantum Computer

One of the most serious difficulties in the quantum gravity research is analysis of the semiclassical regime of the theory... more»

Gravity/Entanglement Duality

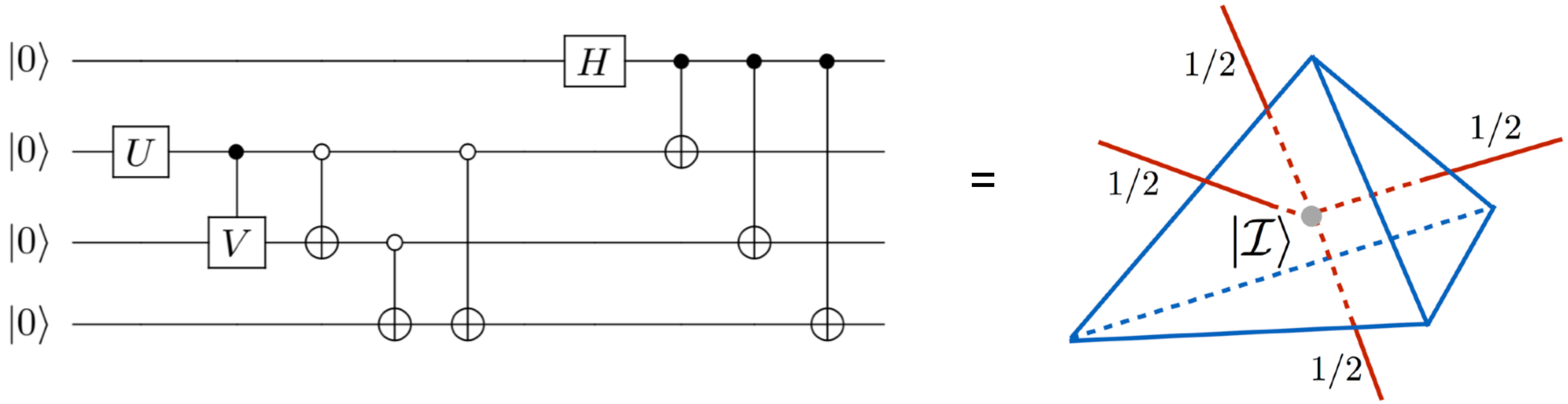
Quantum entanglement is one the most amazing phenomena in quantum mechanics... more»

Quantum Space Technologies

Quantum and gravity do not only meet within fundamental physics at the Planck scale... more»

Motivation I

Quantum simulations of the Planck scale physics.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U
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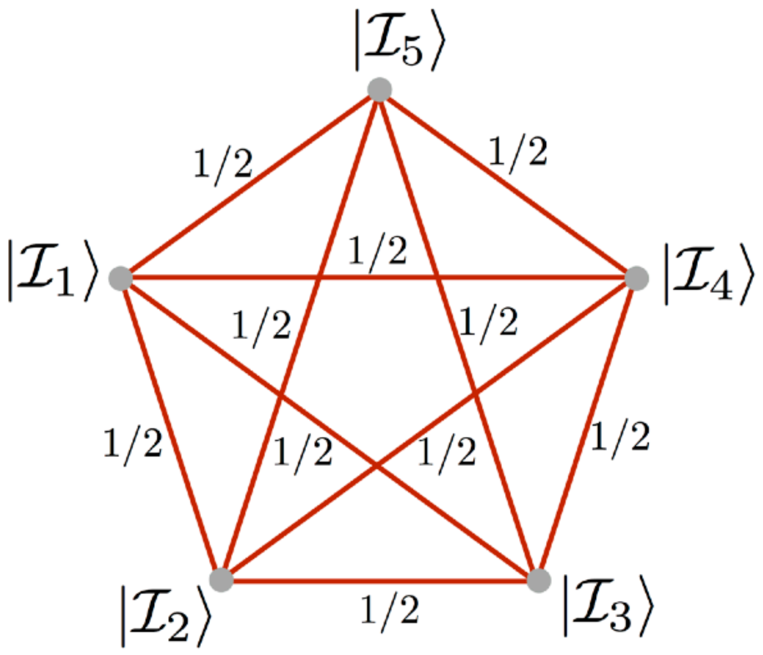
=
$1/2$
$1/2$
$1/2$
$1/2$
$|\mathcal{I}\rangle$

Spin network states $|\Psi\rangle = \bigotimes_i |\mathcal{I}_i\rangle$ **are product states (no entanglement).**

What are the geometric states?

Are they e.g. $|W\rangle = \bigotimes_{l=1}^{10} |\mathcal{E}_l\rangle$

with $|\mathcal{E}_l\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right)$ **?**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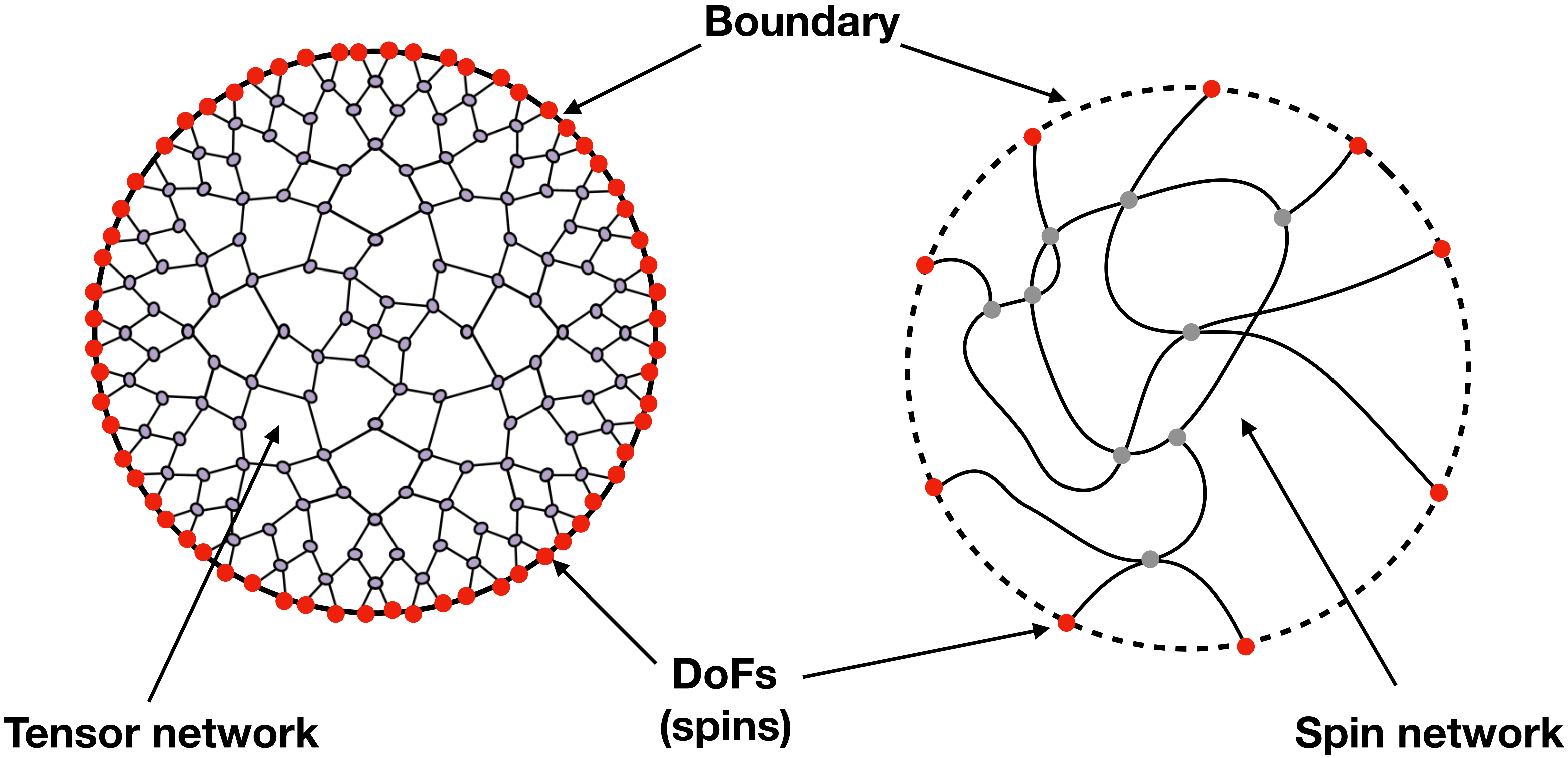
Motivation 2

AdS/CFT

(Gravity/Entanglement)

LQG



Boundary

DoFs
(spins)

Tensor network

Spin network

Is the similarity only apparent?

SU(2) gauge fields

We are focused on SU(2) gauge fields because of their role in describing gravity - Ashtekar variables used to introduce LQG.

SU(2) symmetry is related to the local SO(3) rotations of triads.

$$q^{ab} = \frac{E^a_i E^b_j \delta_{ij}}{\det E}$$

The canonical pair: $\left\{ E^a_i(x), A^j_b(y) \right\} = \delta^a_b \delta^j_i \delta^{(3)}(x-y)$

where $i, j = 1, 2, 3$ and $a, b = 1, 2, 3$

We will focus on the SU(2) connection characterized by the 1-form:

$A = A^i_a \tau_i dx^a$ (Ashtekar connection)

The τ_i are generators of the $\mathfrak{su}(2)$ algebra $[\tau_i, \tau_j] = \epsilon_{ijk}\tau_k$

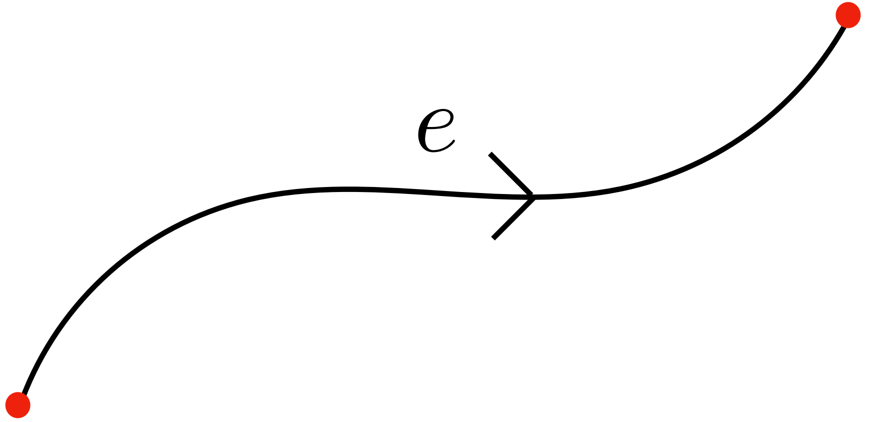
The (first class) Gauss constraint generates local gauge transformations:

$$A_a \to A'_a = U^\dagger A_a U + U^\dagger \partial_a U \qquad U \in SU(2)$$

Holonomies - field theoretical viewpoint

of the SU(2) connection along a path e
are non-local object defined as follows:

$$h_e[A] := \mathcal{P}\exp\int_e A$$



$e(1) = t$ - target

e

$e(0) = s$ - source

path $e : [0,1] \to \Sigma$

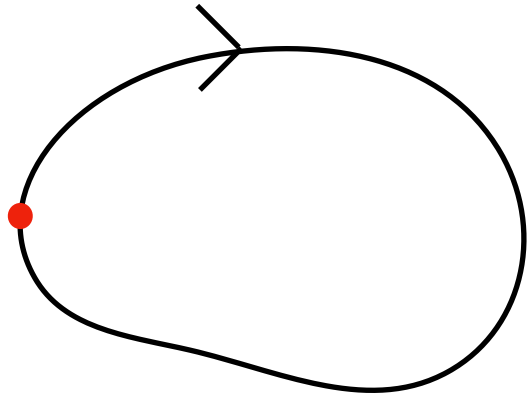
Under the gauge transformation the holonomy transforms as:

$$h_e[A] \to h'_e[A] = U^\dagger(e(0))\, h_e[A]\, U(e(1)) = U_s^\dagger h_e[A] U_t$$

where $U_s := U(e(0))$ and $U_t := U(e(1))$

Gauge invariant objects
- Wilson loops:

$$W_e[A] := \mathrm{tr}(h_e[A])$$



Holonomies - quantum mechanical viewpoint

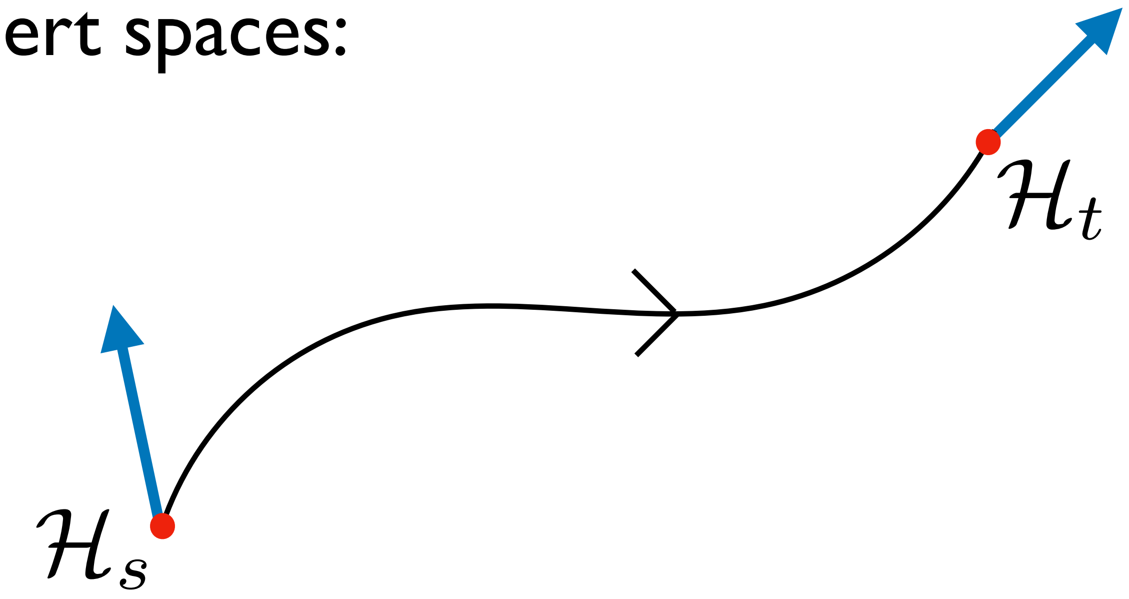
In the fundamental (j=1/2) representation of SU(2), the holonomies are 2x2 SU(2) matrices, which belong to the automorphism group of $\mathbb{C}^2$ (i.e. the space of non-relativistic spinors).

$\mathbb{C}^2$ equipped with the natural scalar product becomes the Hilbert space of a *qubit* system. Pure quantum states correspond to rays in $\mathbb{C}^2$.

The holonomy becomes then an isomorphism (unitary map) between the two 2-dimensional Hilbert spaces:

$$\mathcal{H}_s = \mathrm{span}\,\{|0\rangle_s, |1\rangle_s\}$$

$$\mathcal{H}_t = \mathrm{span}\,\{|0\rangle_t, |1\rangle_t\}$$



For general *j* representation of SU(2), the holonomies are *(2j+1)x(2j+1)* SU(2) matrices, such that $\dim \mathcal{H}_s = 2j+1 = \dim \mathcal{H}_t$

Holonomy as a unitary map

Employing the basis elements of the source and target Hilbert spaces, it is convenient to express an arbitrary holonomy map as:

$$\boxed{h = h_{IJ}|I\rangle_{s\,t}\langle J| \in \mathcal{H}_s \otimes \mathcal{H}_t^*}$$ where $I, J = 0, 1$

The action of this unitary map can be either left-handed or right-handed:

$$h_L : \mathcal{H}_s^* \rightarrow \mathcal{H}_t^* \qquad h_R : \mathcal{H}_t \rightarrow \mathcal{H}_s$$

The Hermitian conjugation of h: $h^\dagger = h^*_{IJ}|J\rangle_{t\,s}\langle I| \in \mathcal{H}_t \otimes \mathcal{H}_s^*$

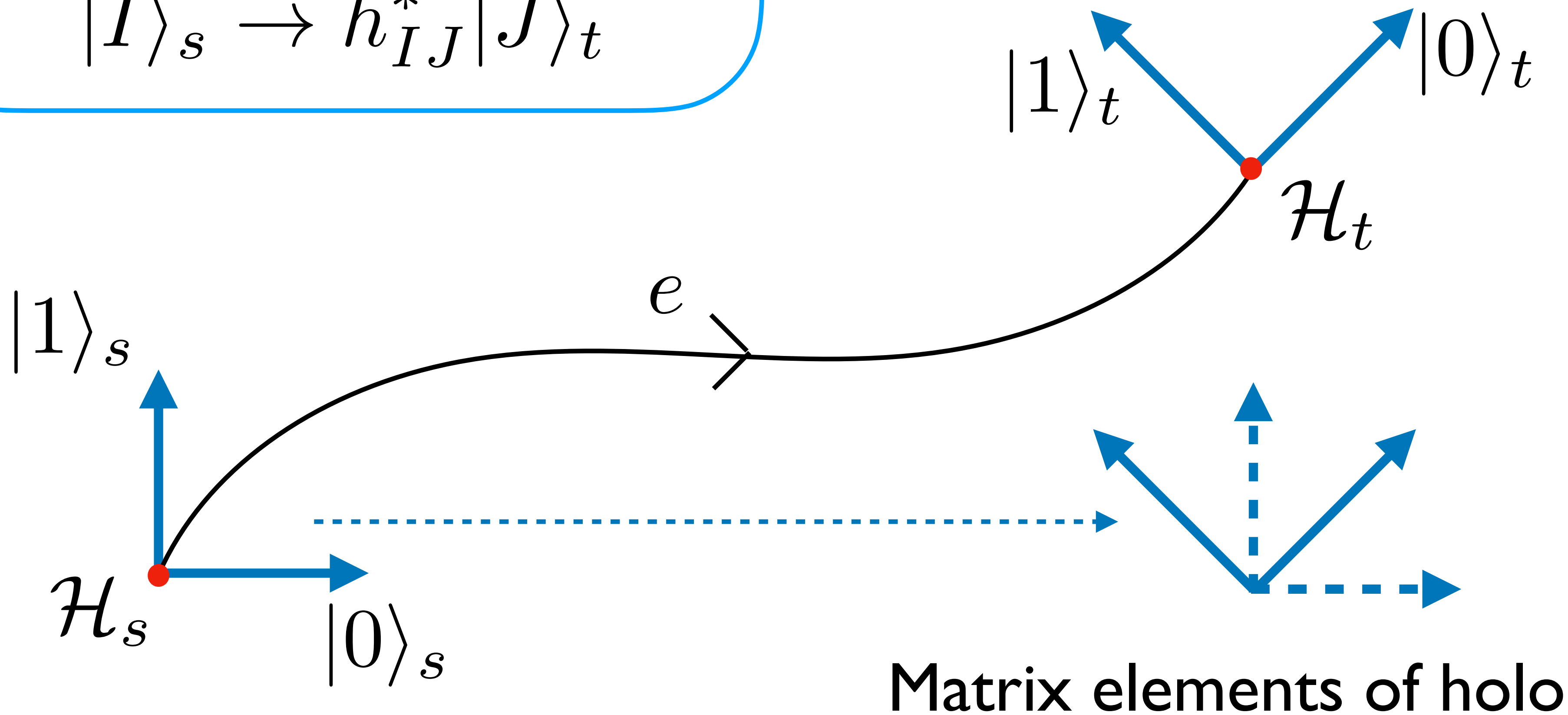
acts as $h_L^\dagger : \mathcal{H}_t^* \rightarrow \mathcal{H}_s^*$ and $h_R^\dagger : \mathcal{H}_s \rightarrow \mathcal{H}_t$

Example: A basis state ${}_s\langle K| \in \mathcal{H}_s^*$ at the point s is mapped into:

$${}_s\langle K|\, h = h_{IJ\,s}\langle K|I\rangle_{s\,t}\langle J| = h_{KJ\,t}\langle J| \in \mathcal{H}_t^*$$ at the point t.

Mapping of the basis states:

$$|I\rangle_s \to h^*_{IJ}|J\rangle_t$$

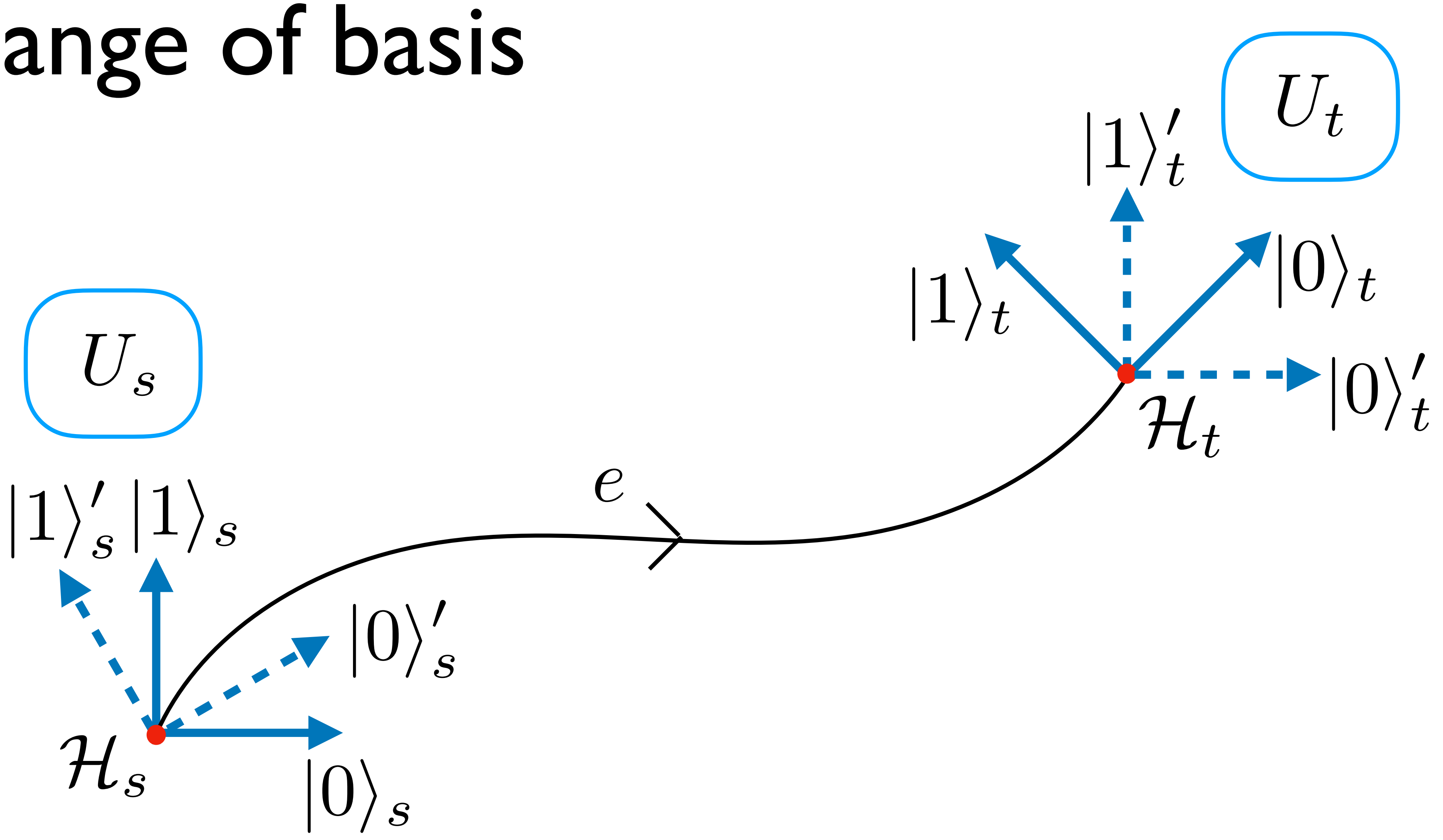

$|1\rangle_t$
$|0\rangle_t$
$\mathcal{H}_t$
e
$|1\rangle_s$
$\mathcal{H}_s$
$|0\rangle_s$

Matrix elements of holonomies
are the coefficients of decomposition
of one basis into the other

Physical interpretation: gravitational holonomies describe displacement of a quantum system from point s to point t in the gauge field:

$$|\psi\rangle \to \mathcal{P}e^{-\int_e A}|\psi\rangle$$

Change of basis


U_t
$|1\rangle'_t$
$|1\rangle_t$
$|0\rangle_t$
$|0\rangle'_t$
$\mathcal{H}_t$
e
U_s
$|1\rangle'_s$
$|1\rangle_s$
$|0\rangle'_s$
$\mathcal{H}_s$
$|0\rangle_s$

Physics does not depend on the choince of basis.

One can perform unitary transformation: $|I\rangle' = U|I\rangle$

or using components of the unitary matrix $|I\rangle' = U_{JI}|J\rangle$

How the holonomy transforms under the change of bases?

$$\begin{tikzcd}
\mathcal{H}_s \arrow[d, "U_s"'] & \mathcal{H}_t \arrow[l, "h"'] \arrow[d, "U_t"] \\
\mathcal{H}'_s & \mathcal{H}'_t \arrow[l, "h'"]
\end{tikzcd}$$

The action of holonomy is preserved under the transformation of bases if

$$h'_{IJ}|I\rangle'_{s\,t}\langle J|' = U_{s,KI}h'_{IJ}U^{\dagger}_{t,JL}|K\rangle_{s\,t}\langle L| = h_{IJ}|I\rangle_{s\,t}\langle J|$$

which leads to the transformation rule: $\boxed{h \to h' = U_s^{\dagger} h U_t}$

It is clear that the change of h under unitary transformations in the source and target spaces is equivalent to the action of a SU(2) gauge transformation.

States in $\mathcal{H}_s \otimes \mathcal{H}_t$

What are the states in $\mathcal{H}_s \otimes \mathcal{H}_t$?

Let us consider:

$$\boxed{|\Psi\rangle := \frac{1}{\sqrt{2}} h^*_{IJ} |I\rangle_s |J\rangle_t \in \mathcal{H}_s \otimes \mathcal{H}_t}$$

where h_{IJ} are matrix components of the SU(2) matrix.

This way of expressing 2-particle states is known in quantum information community (e.g. Bengtsson & Życzkowski, 2017)

The state can be used to introduce anti-linear map & relation to quantum teleportation (Czech, Lamprou & Susskind, 2018; Czech, De Boer, Ge & Lamprou, 2019)

Improved analysis, gravity, networks, etc. (JM & Trześniewski, 2019)

The state $|\Psi\rangle := \frac{1}{\sqrt{2}} h^*_{IJ} |I\rangle_s |J\rangle_t \in \mathcal{H}_s \otimes \mathcal{H}_t$

is a **maximally entangled** state. The density matrix:

$$\hat{\rho} = |\Psi\rangle\langle\Psi| = \frac{h^*_{IJ} h_{KL}}{2} (|I\rangle_{ss}\langle K|)(|J\rangle_{tt}\langle L|)$$

Unitarity of h:

$$h_{IK} h^\dagger_{KJ} = \delta_{IJ}$$

The reduced density matrix:

$$\hat{\rho}_s := \mathrm{tr}_t(\hat{\rho}) = \frac{\boxed{h^*_{IJ} h^T_{JK}}}{2} (|I\rangle_{ss}\langle K|) = \frac{1}{2}\hat{I}$$

The same for

$$\hat{\rho}_t := \mathrm{tr}_s(\hat{\rho})$$

The mutual information is maximal:

$$\boxed{I(s:t) = S(\rho_s) + S(\rho_t) - S(\rho) = 2\ln 2}$$

$$S(\rho_{s,t}) = -\mathrm{tr}(\rho_{s,t} \ln \rho_{s,t}) = \ln 2$$

$S(\rho) = 0$ (pure state)

Anti-linear map

Equivalently to the case of holonomy, one can the following map:

$$\boxed{\mathcal{H}_s^* \ni {}_s\langle I| \to \left[\sqrt{2}\,|\Psi\rangle \circ C\right] \left({}_s\langle I|\right) = h^*_{IJ}|J\rangle_t \in \mathcal{H}_t}$$

where C is the complex conjugation operation.

Change of bases leads to

$$\begin{aligned} {}_s\langle I|' \to \left[\sqrt{2}\,|\Psi\rangle \circ C\right] \left({}_s\langle I|'\right) &= \sqrt{2}\left(U^*_{s,JI}\right)^* {}_s\langle J|\Psi\rangle \\ &= U^T_{s,IJ} h^*_{JL}|L\rangle_t = U^T_{s,IJ} h^*_{JL} U^*_{t,LM}|M\rangle'_t \end{aligned}$$

which leads to the following transformation rule:

$$\boxed{h_{JM} \to h'_{JM} = U^\dagger_{s,IJ} h_{JL} U_{t,LM}}$$

The map is equivalent to the SU(2) gauge transformation.

Gauge transformations from entanglement

Every maximally entangled 2-particle state can be associated with SU(2) holonomy.

For qubits:

$$|\Psi\rangle = \frac{1}{\sqrt{2}} h^*_{IJ} |I\rangle_s |J\rangle_t \in \mathcal{H}_s \otimes \mathcal{H}_t \quad \text{where} \quad I, J = 0, 1$$

In general:

$$|\Psi\rangle = \frac{1}{\sqrt{2j+1}} h^*_{IJ} |I\rangle_s |J\rangle_t \quad \text{where} \quad I, J = 0, \ldots, 2j$$

The h_{IJ} are components of SU(2) matrix, which under the change

of bases transform as $h_{JM} \to h'_{JM} = U^\dagger_{s,IJ} h_{JL} U_{t,LM}$

This holds under assumption that $|\Psi\rangle$ defines the unitary map.

Example: Maximally entangled singlet state:

$$|\Psi\rangle = |\Phi^-\rangle := \frac{1}{\sqrt{2}} \left(|0\rangle_s|1\rangle_t - |1\rangle_s|0\rangle_t\right)$$

The state can be written as

$$|\Psi\rangle = \frac{1}{\sqrt{2}} h^*_{IJ}|I\rangle_s|J\rangle_t \in \mathcal{H}_s \otimes \mathcal{H}_t$$

where $h = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -i\sigma_y = e^{-i\frac{\pi}{2}\sigma_y}$

For simplicity, let us consider a one-parameter family of rotations:

$$U(\theta) = e^{-i\theta\sigma_y} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

The angle changes continuously between s and t, such that:

$$\theta_s := \theta(e(0) = s) \qquad \theta_t := \theta(e(1) = t)$$

Under the change of bases, the map h transforms as follows

$$h \to h' = U_s^\dagger h U_t = e^{-i\frac{\pi}{2}\sigma_y - i(\theta_t - \theta_s)\sigma_y}$$
$$= he^{-i\sigma_y \int_e \partial_a \theta dx^a}$$

One can, therefore, conclude that the matrix h can be written as exponentiation of a certain integral, such that its integrand under the change of bases transforms as:

$$\boxed{\text{integrand} \to \text{integrand} - i\partial_a\theta}$$

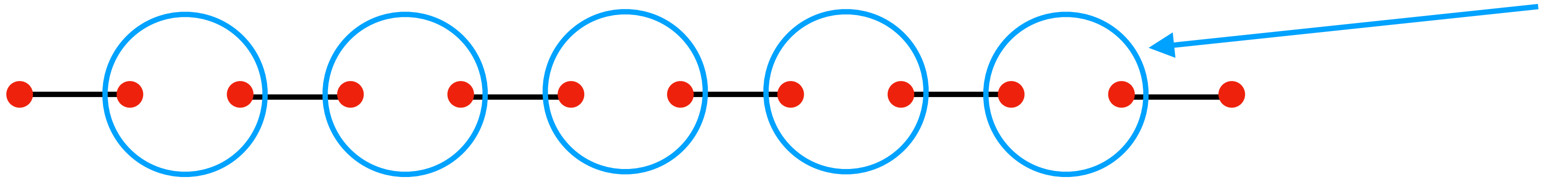
The integrand is what we can call a gauge field.

The U(1) gauge symmetry is a remnant of the SU(2) gauge symmetry restricted to the case of a one-parameter family of transformations.

How to understand this?

Let us consider a discrete system of quantum DoFs, e.g a spin chain

The DoFs form maximally entangled pairs, e.g.



Possible additional constraint (projection)

Physical quantities have to be unaffected by local change of bases

The change of bases lead to the characteristics transformation law of the holonomies representing the maximally entangled pairs. Transformations laws of the local DoFs must compensate the factors resulting from the change of bases (precursors of gauge transformations).

In the continuous limit the field theoretical description is recovered.

CONJECTURE: The gauge field theories are continuous approximations to discrete quantum system characterized by maximally entangled pairs.

Spatial entanglement from holonomies

Let us consider a state of two qubits at the source:

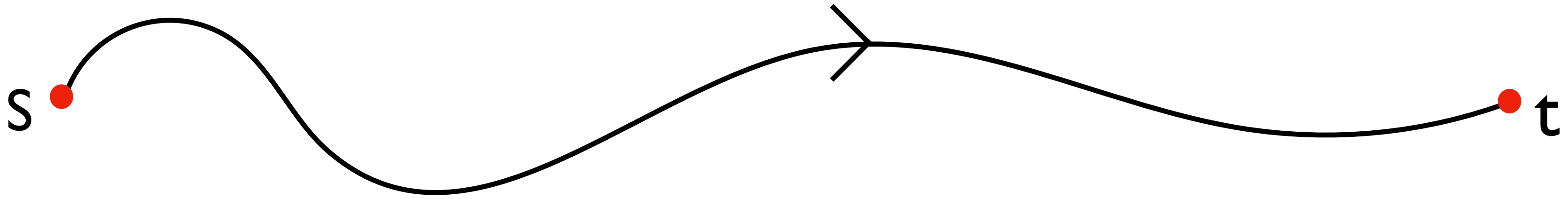
$$|\phi_s\rangle = S_{KL}|K\rangle_s|L\rangle_s \in \mathcal{H}_s \otimes \mathcal{H}_s$$

Take one of the qubits and move it to t, which is given by the unitary map

$$\mathbb{I} \otimes h_R^\dagger : \mathcal{H}_s \otimes \mathcal{H}_s \to \mathcal{H}_s \otimes \mathcal{H}_t$$

$$\begin{aligned} |\phi_s\rangle &\to S_{KL}|K\rangle_s h^\dagger|L\rangle_s = S_{KL}h^*_{LJ}|K\rangle_s|J\rangle_t \\ &\equiv C_{KJ}|K\rangle_s|J\rangle_t =: |\phi_{st}\rangle \in \mathcal{H}_s \otimes \mathcal{H}_t \,. \end{aligned}$$

$$C_{KJ} = S_{KL}h^*_{LJ}$$



The state $|\phi_{st}\rangle$ is equal to $|\Psi\rangle := \frac{1}{\sqrt{2}} h^*_{IJ}|I\rangle_s|J\rangle_t \in \mathcal{H}_s \otimes \mathcal{H}_t$

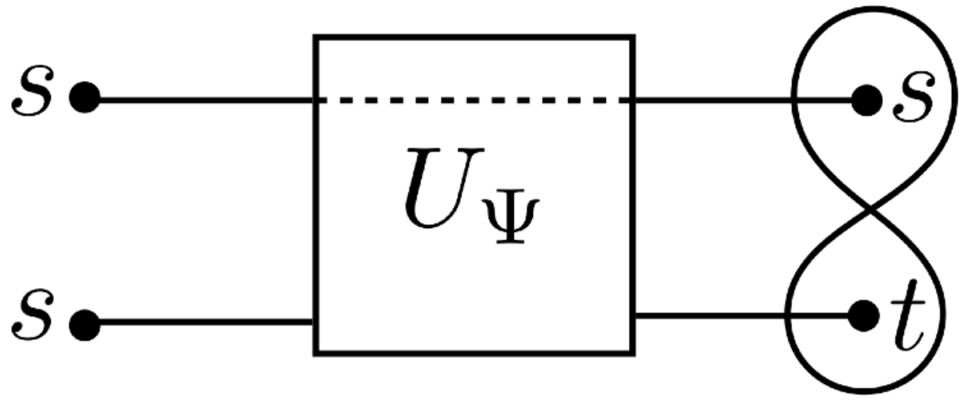
if $S_{KL} = \frac{1}{\sqrt{2}}\delta_{KL}$ so $|\phi_s\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle_s|0\rangle_s + |1\rangle_s|1\rangle_s\right) =: |\Phi^+\rangle$

In consequence $\boxed{|\Psi\rangle = \left(\mathbb{I} \otimes h^\dagger_R\right)|\Phi^+\rangle}$

Using the fact that $|\Phi^+\rangle = \mathrm{CNOT} \circ (\mathrm{H} \otimes \mathbb{I})\,|0\rangle_s|0\rangle_s$

we can define the map

$$\boxed{U_\Psi := (\mathbb{I} \otimes h^\dagger_R)(\mathrm{CNOT})(\mathrm{H} \otimes \mathbb{I})}$$

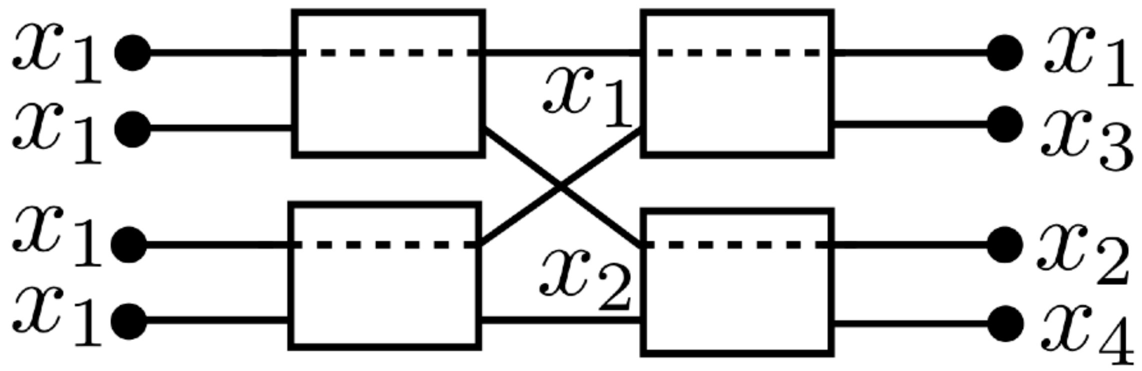


which generates the maximally entangled state from $|0\rangle_s|0\rangle_s$

The maximally entangled pair at two different points can be produced by the action of the map at the pair initially at the same location. Initial contact is needed to generate entanglement.

The maps U_Ψ can be employed to construct circuits.

Exemplary circuit for 4 particles:


x_1
x_1
x_1
x_1
x_1
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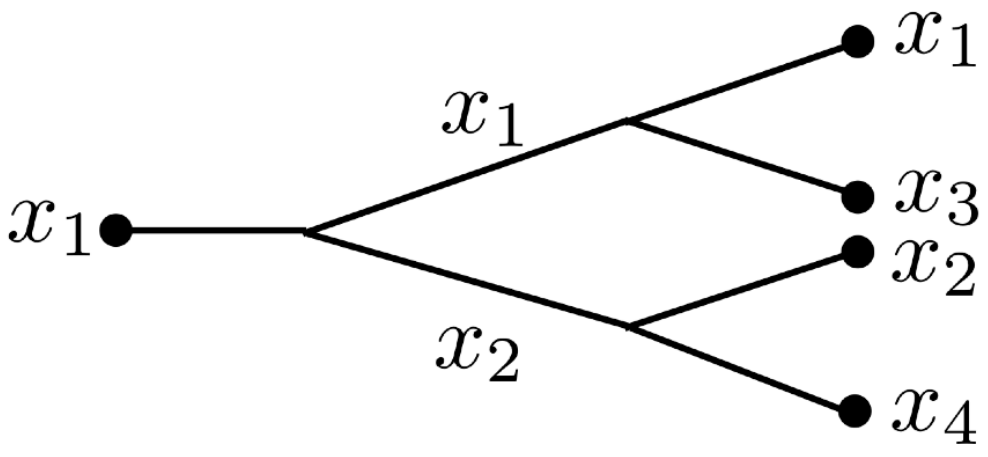
x_3
x_2
x_4

The corresponding splitting a single point into four:


x_1
x_1
x_2
x_1
x_3
x_2
x_4

Is that a way of creating space?

Entangled states of spin networks

The considerations indicate that every holonomy can be associated with a maximally entangled state. This supports gluing polyhedra with entanglement.

A link of a spin network can be associated with the state:

$$|\mathcal{E}_l\rangle := \frac{1}{\sqrt{2}} h^{*}_{IJ} |I\rangle_{s,l} |J\rangle_{t,l} \in \mathcal{H}_{s,l} \otimes \mathcal{H}_{t,l}$$

Next, we produce the following tensor product: $|\mathcal{E}\rangle = \bigotimes_{l} |\mathcal{E}_l\rangle$

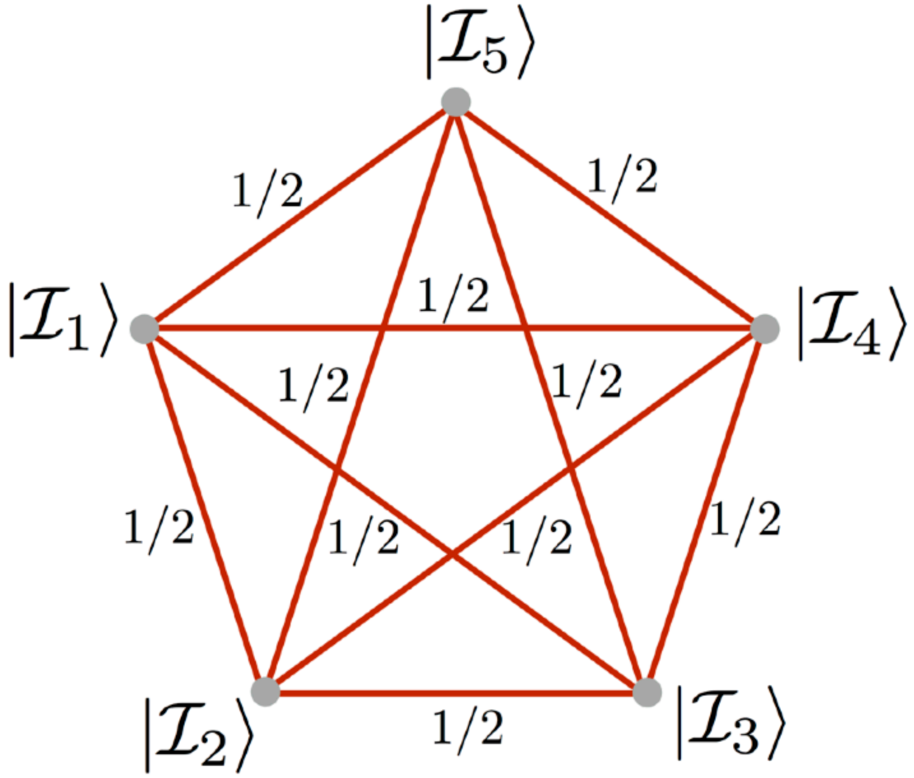
Subsequently, the Gauss constraint is satisfied by projecting the state onto the invariant subspaces at the nodes.

$$\boxed{|\mathrm{MESN}\rangle := \hat{P}_G \bigotimes_{l} |\mathcal{E}_l\rangle}$$

Maximally Entangled
Spin Network

The state is a superposition of spin network states.

Example: Pentagram spin network with the links corresponding to the fundamental representation of SU(2).


$|\mathcal{I}_5\rangle$
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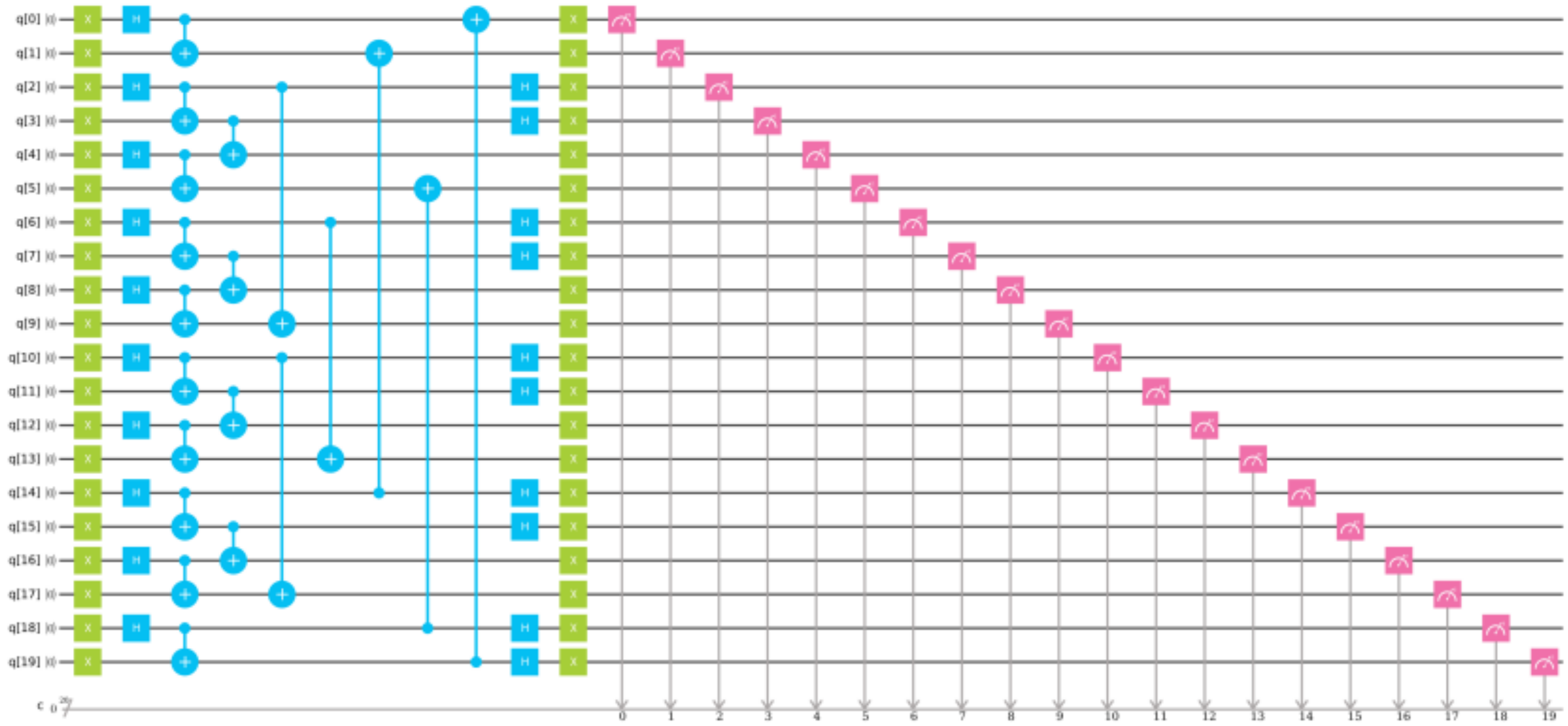
$$|\mathrm{MESN}\rangle = \hat{P}_G \bigotimes_{l=1}^{10} |\mathcal{E}_l\rangle$$

where $$|\mathcal{E}_l\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$


q[0]
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q[10]
q[11]
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q[14]
q[15]
q[16]
q[17]
q[18]
q[19]
c
X
H

A circuit enabling to determine:

$$|\langle \mathrm{MESN}|\mathcal{I}\rangle|^2$$

where

$$|\mathcal{I}\rangle = \bigotimes_{n=1}^{5} |\mathcal{I}_n\rangle$$

(JM, 2019)

Furthermore, components of a holonomy are obtained by projecting the state

$$|\Psi\rangle = \frac{1}{\sqrt{2}} h^*_{IJ} |I\rangle_s |J\rangle_t \in \mathcal{H}_s \otimes \mathcal{H}_t$$

onto the basis states:

$$\boxed{h_{IJ} = \sqrt{2}\left(\left({}_s\langle I|_t\langle J|\right)|\Psi\rangle\right)^* = \sqrt{2}\langle\Psi|\left(|I\rangle_s |J\rangle_t\right)}$$

In consequence, the Wilson loops are:

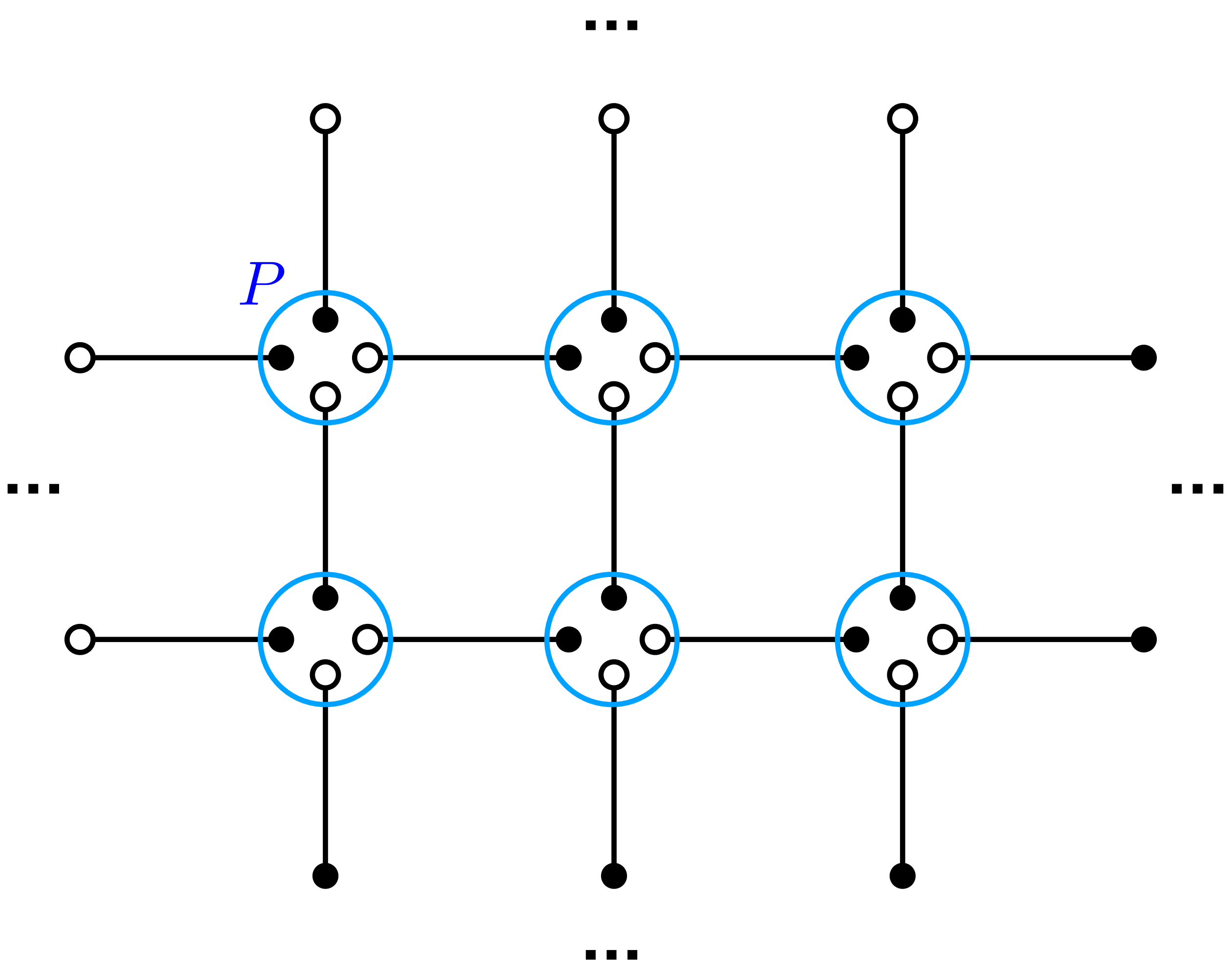
$$\boxed{W_e[h] = \sum_I h_{II} = \sqrt{2}\sum_I \langle\Psi|\left(|I\rangle|I\rangle\right)}$$

Using recouping theory of the SU(2) group, spin networks can be expressed in terms of products and superpositions of the Wilson loops and in consequence as the function of $|\Psi\rangle$.

Relation with tensor networks

Tensor network

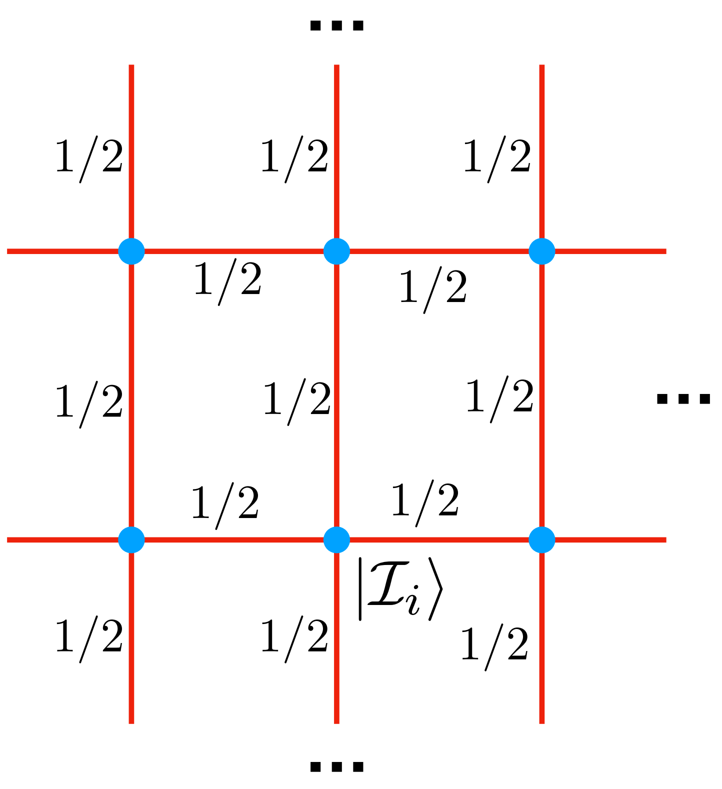
(PEPS - Projected Entangled Pair States)



= $\sum \dots$

Spin network

(Ising type MESN with j=1/2)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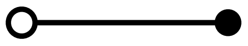
$|\mathcal{I}_i\rangle$
...
...

 = $|\mathcal{E}_l\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$

$|\mathcal{I}_i\rangle$ - intertwiner qubit

P - projection onto spin-0 subspace of four spins (dim = 2)

$$|\Psi\rangle = \sum_{1,2,\dots,n} a_{12\dots n} |\mathcal{I}_1\rangle|\mathcal{I}_2\rangle \dots |\mathcal{I}_n\rangle$$

Thank you!

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